INTRODUCTION TO BRUHAT-TITS BUILDINGS
OCTOBER 2013 AT THE UNIVERSITY OF CHICAGO

DAVE WITTE MORRIS, UNIVERSITY OF LETHBRIDGE
http://people.uleth.ca/~dave.morris/

Abstract. This minicourse will present a very brief introduction to Bruhat-Tits buildings, and describe several applications in group theory that are of geometric or topological interest. The simplest buildings are trees, and can be used to prove Ihara's Theorem that every torsion-free discrete subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ is free. In general, these buildings are metric spaces of non-positive curvature that provide p-adic analogues of Riemannian symmetric spaces.

Lecture 1. Naive applications of Bruhat-Tits buildings

Let Γ be a countably infinite group (with the discrete topology). By definition, the cohomology of Γ is determined by the topology of any $K(\Gamma, 1)$ space:

$$H^*(\Gamma; M) = H^*\big(K(\Gamma, 1); M\big) = H^*\big(X/\Gamma; M\big),$$

where X is any contractible space on which Γ acts properly discontinuously (assuming, for simplicity, that Γ is torsion free[1]). Therefore, to study this cohomology (and for other purposes), it is very helpful to have a nice space that can be used as X.

When Γ is linear (i.e., isomorphic to a subgroup of $\mathrm{SL}_n(\mathbb{C})$, for some n), we will see that such a space can often be constructed from certain simplicial complexes known as "Bruhat-Tits buildings."[2] For example, the following theorem is an easy consequence of basic facts about these objects.

Theorem (Serre, 1971 [16, Thm. 5]). *Every finitely generated, torsion-free subgroup Γ of $\mathrm{SL}_n(\mathbb{Q})$ has finite cohomological dimension. That is, there is some k_0, such that $H^k(\Gamma; M) = 0$ for all $k > k_0$ and every Γ-module M.*

It is necessary to assume that Γ has no torsion in Serre's Theorem, because every group with a nontrivial element of finite order has infinite cohomological dimension [5, Cor. 8.2.5, p. 187]. On the other hand, a well-known theorem of

[1] Since the action is properly discontinuous, the stabilizer of every point is finite. Since Γ is torsion free, this implies that the stabilizers are trivial. I.e., the action is free. So $\Gamma = \pi_1(X/\Gamma)$ and, since X is contractible, X/Γ is a $K(\Gamma, 1)$.

[2] **Warning.** *Bruhat-Tits buildings* (also known as "Euclidean" buildings, or "affine" buildings) are not the same as *Tits buildings* (also known as "spherical buildings"). However, they are related by the fact that the link of a vertex in a Bruhat-Tits building is a Tits building. (Also, a Bruhat-Tits building can be compactified by adding a sphere at ∞, and this boundary sphere has the structure of a Tits building.) These lectures will not make any attempt to describe the theory or applications of any type of building other than Bruhat-Tits buildings.

Selberg tells us that every finitely generated subgroup of $\mathrm{SL}_n(\mathbb{C})$ has a torsion-free subgroup of finite index (see [14, Thm. 6.11], for example). Therefore, we can restate the theorem as follows:

Theorem (Serre)**.** *Every finitely generated subgroup Γ of $\mathrm{SL}_n(\mathbb{Q})$ has finite virtual cohomological dimension. That is, there is some k_0, and some finite-index subgroup Γ' of Γ, such that $H^k(\Gamma'; M) = 0$ for all $k > k_0$ and every Γ'-module M.*

If we make the stronger assumption that the matrix entries are integers, not just rational, then there is no need for buildings in the proof, because it suffices to use only a symmetric space. Namely, let

$$\Gamma_\infty = \mathrm{SL}_n(\mathbb{Z}), \quad G_\infty = \mathrm{SL}_n(\mathbb{R}), \quad K_\infty = \mathrm{SO}(n),$$

so Γ_∞ is a discrete subgroup of the Lie group G_∞, and K_∞ is a (maximal) compact subgroup of G_∞. It is therefore obvious that Γ_∞ acts properly discontinuously on the "symmetric space" $X_\infty = G_\infty/K_\infty$. As a consequence of the "Iwasawa decomposition" $G_\infty = K_\infty A_\infty N_\infty$, it is well known that X_∞ is contractible. Since X_∞ is a manifold (hence, finite-dimensional), we conclude that every torsion-free subgroup of Γ_∞ has finite cohomological dimension. More precisely, we may take

$$k_0 = \dim X_\infty = \dim G_\infty - \dim K_\infty = (n^2 - 1) - \frac{n(n-1)}{2} = \frac{n(n+1)}{2} - 1.$$

Bruhat-Tits buildings are analogues of symmetric spaces that are constructed by using p-adic numbers.[3]

Notation. Let
- p be a prime number,
- $\mathbb{Q}_p$ be the field of p-adic numbers, and
- $G_p = \mathrm{SL}_n(\mathbb{Q}_p)$.

(More generally, G_p can be any semisimple algebraic group over $\mathbb{Q}_p$.)

Recall.
- p-adic norm on $\mathbb{Q}$: $\|a/b\|_p \approx 0$ if a is divisible by a large power of p (and $p \nmid b$).
- $\mathbb{Q}_p$ = completion of $\mathbb{Q}$ under this norm = {limits of Cauchy sequences}
$$= \left\{ \sum_{i=v}^{\infty} a_i p^i \;\middle|\; \begin{array}{c} v \in \mathbb{Z}, \\ a_i \in \{0,1,2,\ldots,p-1\} \end{array} \right\}. \quad \text{(totally disconnected)}$$
- $\left|\left|\sum_{i=v}^{\infty} a_i p^i\right|\right|_p = p^{-v}$ if $a_v \neq 0$,
  - $\|xy\|_p = \|x\|_p \, \|y\|_p$,

[3]We consider only $\mathbb{Q}_p$ for simplicity, but, for those with the appropriate background, the theory generalizes in a straightforward way when $\mathbb{Q}_p$ is replaced by any finite extension Q_v of $\mathbb{Q}_p$, such as $Q_v = \mathbb{Q}_p[\sqrt{3}]$ (if 3 does not have a square root in $\mathbb{Q}_p$). In fact, the local field Q_v can have finite characteristic.

- ∘ $\|x+y\|_p \le \max\{\|x\|_p, \|y\|_p\}$ (“ultrametric inequality”)
- $\mathbb{Q}_p^\times = \{p^k\} \times \{\, x \mid \|x\|_p = 1\,\} \cong \mathbb{Z} \times \text{compact}$

Note that

$$G_p \supset \{\text{diagonal matrices}\} \cong (\mathbb{Q}_p^\times)^{n-1} \cong \mathbb{Z}^{n-1} \times \text{compact} \supset \mathbb{Z}^{n-1}.$$

Since $H^{n-1}(\mathbb{Z}^{n-1};\mathbb{R}) = H^{n-1}(\mathbb{T}^{n-1};\mathbb{R}) = \mathbb{R} \neq 0$, this implies that it is not possible for G_p to act properly on any contractible simplicial complex of dimension less than $n-1$. Surprisingly, this bound can be attained:

Theorem (Goldman-Iwahori,[4] Iwahori-Matsumoto, Bruhat-Tits, ~1965)**.** *There is a simplicial complex X_p, called the **Bruhat-Tits building of G_p**, such that:*

1. *there is a continuous action of G_p on X_p, by simplicial automorphisms, and this action is proper,*
2. *X_p is contractible,*
3. *X_p is finite-dimensional (more precisely, $\dim X_p = n-1 = \operatorname{rank}_{\mathbb{Q}_p} G_p$), and*
4. *X_p is locally finite (i.e., each simplex is adjacent to only finitely many other simplices).*

Corollary (Ihara, 1966 [10])**.** *Every torsion-free, discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{Q}_p)$ is free.*

Proof. Since Γ is discrete, it is a closed subgroup of $\mathrm{SL}_2(\mathbb{Q}_p) = G_p$, so (1) tells us that Γ acts properly discontinuously on X_p. Since Γ is torsion-free and X_p is contractible (2), this implies that $\Gamma = \pi_1(X_p/\Gamma)$. However, (3) tells us that

$$\dim(X_p/\Gamma) = \dim X_p = n-1 = 2-1 = 1,$$

which means that the simplicial complex X_p/Γ is 1-dimensional — it is a graph. Therefore, $\pi_1(X_p/\Gamma)$ is a free group. □

Proof of Serre's Theorem. To simplifiy, let us assume p is the only prime that occurs in the denominator of any element of Γ. I.e., $\Gamma \subseteq \mathrm{SL}_n(\mathbb{Z}[1/p]) = \Gamma_p$. The inclusions $\mathbb{Z}[1/p] \hookrightarrow \mathbb{R}$ and $\mathbb{Z}[1/p] \hookrightarrow \mathbb{Q}_p$ provide embeddings of Γ_p in two different groups:

$$\Gamma_p \overset{\varphi_\infty}{\hookrightarrow} G_\infty \quad \text{and} \quad \Gamma_p \overset{\varphi_p}{\hookrightarrow} G_p.$$

Unfortunately, Γ_p is not a discrete subgroup of G_∞ or G_p. (In fact, it is dense in both groups, because $\mathbb{Z}[1/p]$ is dense in both $\mathbb{R}$ and $\mathbb{Q}_p$.) However, we can take the product embedding

$$\varphi\colon \Gamma_p \xrightarrow{\varphi_\infty \times \varphi_p} G_\infty \times G_p \quad \text{defined by} \quad \varphi(\gamma) = \big(\varphi_\infty(\gamma), \varphi_p(\gamma)\big).$$

[4] Goldman-Iwahori provided an explicit construction of the space X_p for $G_p = \mathrm{SL}_n(\mathbb{Q}_p)$ (our standing assumption). This was generalized to split groups by Iwahori-Matsumoto, and to all reductive p-adic groups by Bruhat-Tits.

The image of this embedding is discrete. (Exercise: $\mathbb{Z}[1/p]$ is discrete in $\mathbb{R} \times \mathbb{Q}_p$.)

Since G_∞ and G_p act properly on X_∞ and X_p, respectively, the product $G_\infty \times G_p$ acts properly on the Cartesian product $X = X_\infty \times X_p$. So the discrete subgroup $\varphi(\Gamma_p)$ acts properly discontinuously on this space, which is contractible and finite-dimensional (since each factor of the two factors is contractible and has finite dimension). Therefore, every torsion-free subgroup of Γ_p has finite cohomological dimension. $\square$

Exercise. $\mathbb{Z}[1/p_1, \ldots, 1/p_k]$ is a discrete subring of $\mathbb{R} \oplus \bigoplus_{i=1}^k \mathbb{Q}_{p_i}$.

Remark. The above proof made a simplifying assumption. In general, the product defining X may need to include several Bruhat-Tits buildings. Namely, if we let $\pi(\Gamma)$ be the set of distinct primes that occur in the denominators of the matrix entries of the elements of any (finite) generating set for Γ, then the product will need to include a Bruhat-Tits building for each $p \in \pi(\Gamma)$:

$$X = X_\infty \times \prod_{p \in \pi(\Gamma)} X_p.$$

Since Γ acts properly discontinuously on this contractible space, the above proof shows that we may take

$$k_0 = \dim X = \frac{n(n+1)}{2} \; + \; \#\pi(\Gamma) \cdot (n-1) \; - \; 1.$$

The statement of Serre's Theorem requires the matrix entries of all the elements of Γ to be rational numbers. However, a standard "Restriction of Scalars" argument allows this hypothesis to be weakened:

Corollary. *Suppose Γ is a finitely generated, torsion-free subgroup of $\mathrm{SL}_n(\mathbb{C})$. If all the matrix entries of the elements of Γ are algebraic numbers, then Γ has finite cohomological dimension.*

Proof. Let F be the field extension of $\mathbb{Q}$ that is generated by the matrix entries of the elements of Γ. This is a finite extension of some degree d, since Γ is finitely generated. By definition of F, we have $\Gamma \subseteq \mathrm{SL}_n(F)$. Since $F^n \cong \mathbb{Q}^{dn}$ as a vector space over $\mathbb{Q}$, we can identify Γ with a subgroup of $\mathrm{SL}_{dn}(\mathbb{Q})$, so Serre's Theorem applies. $\square$

The following example provides a counterexample if we do not require the matrix entries to be algebraic:

Example. If ξ is any transcendental number, and

$$\Gamma = \left\{ \begin{bmatrix} \xi^k & x \\ 0 & \xi^{-k} \end{bmatrix} \;\middle|\; k \in \mathbb{Z},\ x \in \mathbb{Z}[\xi] \right\},$$

then Γ is a finitely generated subgroup of $\mathrm{SL}(2, \mathbb{C})$ that is torsion free and has infinite cohomological dimension.

Proof. Let $a = \begin{bmatrix} \xi^k & x \\ 0 & \xi^{-k} \end{bmatrix}$, $b_i = \begin{bmatrix} 1 & \xi^i \\ 0 & 1 \end{bmatrix}$, and $B = \langle\, b_i \mid i \in \mathbb{Z} \,\rangle$.
Then $\Gamma = \langle a \rangle B$, and we have $a b_i a^{-1} = b^{i+2}$, so it is easy to see that $\Gamma = \langle a, b_0, b_1 \rangle$. Therefore, Γ is finitely generated. Also, since $B \cong \mathbb{Z}^\infty$ and $\Gamma/B \cong \mathbb{Z}$ are torsion free, we see that Γ is torsion free.

However, the cohomological dimension of $\mathbb{Z}^r$ is $\dim \mathbb{T}^r = r$, so the subgroup $B \cong \mathbb{Z}^\infty$ obviously has infinite cohomological dimension. Therefore, the cohomological dimension of Γ must be infinite. □

The group Γ in the above example contains upper-triangular subgroups that are isomorphic to $\mathbb{Z}^r$, with r arbitrarily large. The following generalization of Serre's Theorem shows that containing such subgroups is the only obstruction to having finite cohomological dimension.

Definition. Let
$$\mathbb{U}_n = \left\{ \begin{bmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{bmatrix} \right\} \subset \mathrm{SL}_n(\mathbb{C}).$$
A subgroup U of $\mathrm{SL}_n(\mathbb{C})$ is ***unipotent*** if it is conjugate to a subgroup of $\mathbb{U}_n$.

Theorem (Alperin-Shalen, 1982 [2])**.** *A finitely generated, torsion-free subgroup* Γ *of* $\mathrm{SL}_n(\mathbb{C})$ *has infinite cohomological dimension if and only if it contains unipotent, free abelian subgroups of arbitrarily large rank.*

Proof. (⇐) See the proof of the example. This direction is elementary, with no need for Bruhat-Tits buildings.

(⇒) This argument uses more information about the structure of X_p than has been given in this lecture. A sketch can be found in [6, p. 195]. □

It is easy to see that $\Gamma_\infty = \mathrm{SL}_n(\mathbb{Z})$ and $\Gamma_p = \mathrm{SL}\big(n, \mathbb{Z}[1/p]\big)$ are finitely generated, but it is not at all obvious that they are finitely presented. For Γ_∞, this is a consequence of the Borel-Serre compactification:

Theorem (Borel-Serre, 1976 [3])**.** *Adding certain points at infinity to* X_∞ *yields a "partial compactification"* $\overline{X}_\infty^{BS}$*, such that*

(1) Γ_∞ *acts properly discontinuously on* $\overline{X}_\infty^{BS}$*,*
(2) $\overline{X}_\infty^{BS}$ *is contractible, and*
(3) $\overline{X}_\infty^{BS}/\Gamma_\infty$ *is a compact manifold with corners.*

Remark (historical)**.** For $G_\infty = \mathrm{SL}_n(\mathbb{R})$, this construction is attributed to Siegel [17].

Corollary. Γ_∞ *is finitely presented.*

Proof. Assume, for simplicity, that Γ_∞ is torsion free, so it is the fundamental group of $\overline{X}_\infty^{BS}/\Gamma_\infty$. Ignoring a technical issue, let us also assume that $\overline{X}_\infty^{BS}/\Gamma_\infty$ is a CW-complex. Since $\overline{X}_\infty^{BS}/\Gamma_\infty$ is compact, it has only finitely many cells. The fundamental group Γ_∞ is generated by representatives of the finitely many loops in the 1-skeleton of $\overline{X}_\infty^{BS}/\Gamma_\infty$, and a presentation can be obtained by adding a relation for each of the finitely many 2-cells. □

Theorem (Borel-Serre, 1973 [4]). $\mathrm{SL}(n, \mathbb{Z}[1/p])$ *is finitely presented. In other words, it is of type* F_2*.*

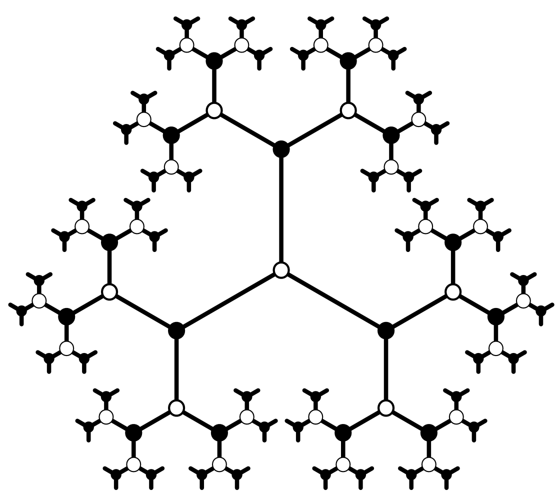
Proof. Note that the stabilizer of any simplex in X_p is a compact, open subgroup of G_p (since G_p acts continuously and properly, by simplicial automorphisms). Using this, it is not difficult to see that Γ_p acts properly discontinuously on the contractible space $\overline{X}_\infty^{BS} \times X_p$. By combining these facts with the compactness of $\overline{X}_\infty^{BS}/\Gamma_\infty$ and X_p/G_p, it is straightforward to show that $(\overline{X}_\infty^{BS} \times X_p)/\Gamma_p$ is compact. □

Remark. To say that a group Γ is of type F_m means that it acts freely and cocompactly on an $(m-1)$-connected CW-complex. Therefore, the proof actually shows that $\mathrm{SL}(n, \mathbb{Z}[1/p])$ is of type F_m for all m. The same can be said with $\mathbb{Z}[1/p_1, 1/p_2, \ldots, 1/p_k]$ in the place of $\mathbb{Z}[1/p]$.

See [6, Chap. 7] for more discussion of the applications in this lecture.

Lecture 2. Examples of Bruhat-Tits buildings

Example (Bruhat-Tits building of $\mathrm{SL}_2(\mathbb{Q}_p)$). We have already seen that the Bruhat-Tits building of $G_p = \mathrm{SL}_2(\mathbb{Q}_p)$ is a tree (since $\dim X_p = n - 1$ in general, and we have $n = 2$ in this case). In fact, X_p turns out to be a regular tree of valence $p+1$: every vertex is in exactly $p+1$ edges. Here is a picture of (a ball in) X_2:



texpreamble Note that any finite geodesic segment has uncountably many different extensions to a (two-way-infinite) geodesic. However, all of these extensions are equivalent under the isometry group. In fact, they are equivalent under G_p:

(BT1) if γ_1 and γ_2 are any two geodesics in X_p, then there exists $g \in G_p$, such that $g(\gamma_1) = \gamma_2$, and g fixes every point of $\gamma_1 \cap \gamma_2$.

This implies that G_p is transitive on the set of edges. However, it turns out that G_p is not transitive on the set of vertices. Instead, the two vertices of any edge are of a different "type" (colored black or white in the figure), and G_p is transitive on the set of vertices of any given type. Thus, if an element g of G_p fixes an edge setwise, then it fixes the edge pointwise. (In the terminology of Serre's book on *Trees*, this means that G_p acts without inversion.)

The group A of diagonal matrices in $G_p = \mathrm{SL}_2(\mathbb{Q}_p)$ fixes a unique geodesic setwise, and acts by translation along this geodesic. (Note the close analogy with the action of $\mathrm{SL}_2(\mathbb{R})$ on the hyperbolic plane. Perhaps we should also point out that, under the identification $A \cong \mathbb{Q}_p^\times \cong \mathbb{Z} \times \text{compact}$, the group $\mathbb{Z}$ acts by translations on the geodesic, and the compact group acts trivially on this geodesic.) Since the maximal split tori in G_p are all conjugate, and the geodesics in X_p are all equivalent under G_p, this implies that there is a natural 1–1 correspondence

$$\{\text{split tori}\} \leftrightarrow \{\text{geodesics}\} \quad \text{defined by} \quad A \leftrightarrow \text{geodesic fixed by } A.$$

Remark. The nonuniqueness of geodesic extensions (or "branching"), can also be seen in the coarse geometry of the hyperbolic plane. Namely, if two geodesics are asymptotic at $-\infty$, but not at $+\infty$, then, looking from a far distance, they are indistiguishable from two geodesics that are equal for all time in the past, but are heading off in two completely different directions in the future.

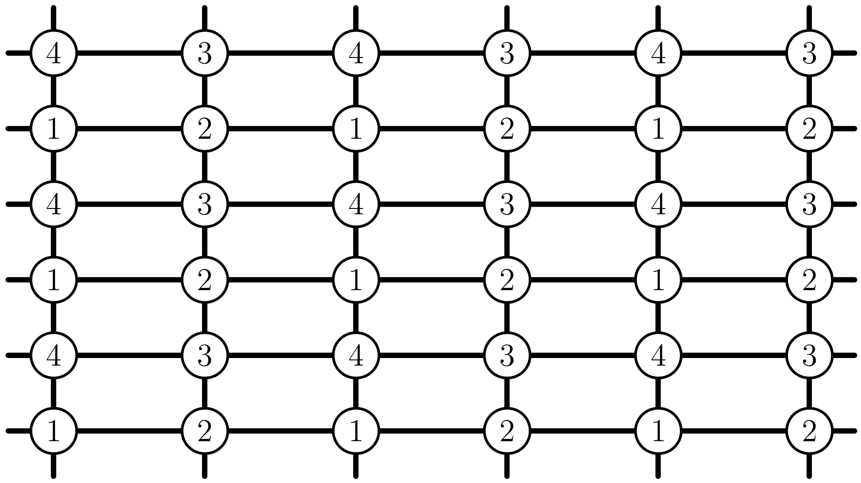
In these lectures, we are assuming that G_p is the simple group $\mathrm{SL}_n(\mathbb{Q}_p)$, but let us make an exception to discuss the semsimple group $\mathrm{SL}_2(\mathbb{Q}_p)\times\mathrm{SL}_2(\mathbb{Q}_p)$, because it provides an illuminating example.

Example (Bruhat-Tits building of $\mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p)$)**.** In general, the Bruhat-Tits building of a direct product of groups is the Cartesian product of the Bruhat-Tits buildings of the factors. Therefore, the Bruhat-Tits building of $\mathrm{SL}_2(\mathbb{Q}_p)\times\mathrm{SL}_2(\mathbb{Q}_p)$ is the product $T_1 \times T_2$ of two $(p+1)$-regular trees.

The trees T_1 and T_2 each have a unique $\mathrm{SL}_2(\mathbb{Q}_p)$-invariant metric (up to a constant that determines the length of an edge). More generally, the Bruhat-Tits building of $\mathrm{SL}_n(\mathbb{Q}_p)$ (or any other simple group G_p) has a G_p-invariant metric that is canonical, up to a normalizing constant. (We will have more to say about this metric in the final lecture.) However, for a Bruhat-Tits building that is a product, there may be no canonical way to choose the constants on the various factors.[5]

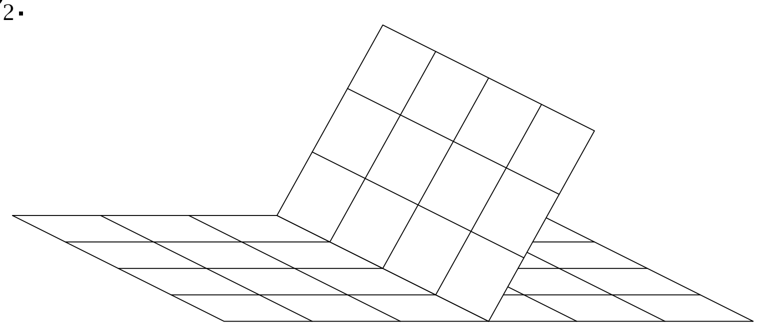
The Cartesian product of a geodesic γ_1 in T_1 with a geodesic γ_2 in T_2 is a subspace of $X_p = T_1 \times T_2$ that is isometric to the Euclidean plane, tiled by rectangles:

$\gamma_1 \times \gamma_2$:



As in any Bruhat-Tits building, G_p is transitive on the vertices of any given type, and each top-dimensional cell[6] has exactly one vertex of each type.

A subset of X_p is a ***flat*** if it is isometric to a Euclidean space $\mathbb{R}^k$. (Note that 1-dimensional flats are geodesics.) Since $\dim X_p = 2$, it is obvious that $F = \gamma_1 \times \gamma_2$ is a maximal flat.[7] Note that if we choose a geodesic γ_1' of T_1 that branches from γ_1 at some vertex v, then $F' = \gamma_1' \times \gamma_2$ is a maximal flat that branches from F along the line $\{v\} \times \gamma_2$.



In fact, since each vertex in γ_1 (or γ_2) is in $p+1$ different edges of T_1 (or T_2, respectively), we see that each edge of X_p is on the boundary of $p+1$ different

[5]For a well-defined normalization that is an analogue of the Killing form for semisimple Lie groups, see the proof of Lemma 4.2 in [D. W. Morris and K. Wortman, Horospherical limit points of S-arithmetic groups, `http://arxiv.org/abs/1309.7113`.

[6]In the theory of Bruhat-Tits buildings, top-dimensional cells are called ***chambers***, but we do not need this terminology.

[7]In the theory of Bruhat-Tits buildings, maximal flats are called ***apartments***, but we are using terminology that is more familiar to geometers. These flats are usually denoted A (for "apartment"), instead of F. Then some other letter, such as R or S or T, must be used for a maximal split torus.

rectangles (2-cells) in X_p, so each of the horizontal or vertical lines in the above figure is a branch locus where maximal flats in X_p branch from each other.

It is easy to see that the isometry group of X_p is **not** transitive on the set of geodesics. (For example, a line of rational slope in $\gamma_1 \times \gamma_2$ cannot be sent to a line of irrational slope, because the set of vertices must be preserved.) However, although it may not be obvious, the isometry group is transitive on the set of maximal flats. In fact, G_p is transitive on this set. (This means that every maximal flat is of the form $\gamma_1' \times \gamma_2'$, where γ_i' is a geodesic in T_i.) Furthermore, (BT1) generalizes in a natural way:

(BT1) if F_1 and F_2 are any two maximal flats in X_p, then there exists $g \in G_p$, such that $g(F_1) = F_2$, and g fixes every point of $F_1 \cap F_2$.

This is a basic property of all Bruhat-Tits buildings — it is a minor modification of an axiom in the theory of buildings.

If we let A_1 and A_2 be the split tori in $\mathrm{SL}_2(\mathbb{Q}_p)$ that fix γ_1 and γ_2, respectively, then it is clear that $A_1 \times A_2$ fixes the maximal flat $\gamma_1 \times \gamma_2$ in $T_1 \times T_2$, and acts on it by translation (via a cococompact group of isometries that is isomorphic to $\mathbb{Z}^2$). We have a natural 1–1 correspondence

$$\{\text{maximal split tori}\} \leftrightarrow \{\text{maximal flats}\} \qquad \text{defined by} \qquad A \leftrightarrow \text{flat fixed by } A,$$

and the same is true in any Bruhat-Tits building.

Remark. It is a basic axiom of ("thick") buildings in general, and Bruhat-Tits buildings in particular, that every simplex[8] of codimension one is a face of at least three different top-dimensional simplexes. This means, in every Bruhat-Tits building, branching of the maximal flats occurs at every codimension-one simplex, as in the above examples.

Exercise. Let F be a maximal flat in a Bruhat-Tits building X_p, and let σ be a codimension-one simplex in F. Show that G_p contains an element g that acts on F via the reflection through the hyperplane that contains σ. (Also, use this observation to verify that the vertices in the depiction of $\gamma_1 \times \gamma_2$ on page 8 have been correctly partitioned into types.)
Hint: Combine the preceding remark with (BT1).

Any two points of X_p are joined by a geodesic. (In fact, it follows from (BT1) that there is a unique geodesic segment joining any two points in X_p [6, p. 152].) Then, since X_p is finite-dimensional, it is obvious that

(BT2) Any two points of X_p are contained in a maximal flat.

[8]If X_p is a product, such as $T_1 \times T_2$, then we should not be using the term "simplex," because the cells in the complex are "polysimplexes" (i.e., Cartesian products of simplexes), not simplexes. We will ignore this technicality, because our interest is in examples such as $G_p = \mathrm{SL}_n(\mathbb{Q}_p)$.

By considering points in the interior of any two given simplexes, we see that if σ_1 and σ_2 are two simplexes of X_p, then there is a maximal flat that contains $\sigma_1 \cup \sigma_2$. This is another axiom in the theory of buildings.

A *Bruhat-Tits building* X_p is a simplicial complex that is constructed in a certain way from a p-adic group G_p (such as $\mathrm{SL}_n(\mathbb{Q}_p)$). Some of the main properties of the construction are summarized in the following definition. For us, the collection $\mathcal{F}$ is the set of maximal flats, but, in the theory of buildings, the elements of $\mathcal{F}$ are called "apartments."

Definition (Tits)**.** An (irreducible) *Euclidean building* is a finite-dimensional simplicial complex X (with a metric), together with a collection $\mathcal{F}$ of subsets of X, satisfying the following axioms:

(1) Each $F \in \mathcal{F}$ is a Coxeter complex in some Euclidean space $\mathbb{R}^d$. This means that the simplicial structure of F is the tiling generated by the hyperplanes of a group generated by reflections.

(2) $\forall F_1, F_2 \in \mathcal{F}$, there is an isometry from F_1 onto F_2 that fixes $F_1 \cap F_2$ pointwise.

(3) Every codimension-one simplex is a face of at least three top-dimensional simplexes.

(4) $\forall x, y \in X,\ \exists F \in \mathcal{F},\ \{x, y\} \subset F$.

We have seen that the Bruhat-Tits building X_p satisfies these properties, so every Bruhat-Tits building is a Euclidean building. Conversely, Tits proved that every Euclidean building of dimension at least four is isomorphic to the Bruhat-Tits building of some group G_p.

Remark. The reason these are called "Euclidean" buildings is that each apartment is a Euclidean space. In a "spherical" building, each apartment is a sphere, and the simplicial structure on the sphere is generated by a group of reflections acting on the sphere. There are also "hyperbolic" buildings, in which each apartment is a hyperbolic space $\mathbb{H}^d$, and the simplicial structure is generated by a group of reflections acting by reflections that are isometries of the hyperbolic metric.

Exercise ([6, Prop. 3.1, p. 85])**.** Let F be a maximal flat in a Bruhat-Tits building X_p, and let σ be a top-dimensional simplex in F. For each $x \in X_p$, choose a maximal flat F' that contains $\sigma \cup \{x\}$, and use (BT1) to choose $g \in G_p$, such that $gF' = F$, and g is the identity on σ. Show that defining $r(x) = gx$ yields a well-defined "retraction" $r \colon X_p \to F$, such that

(1) r is the identity on F,
(2) $d(r(x), y) = d(x, y)$ for all $x \in X_p$ and $y \in \sigma$, and
(3) $r(x)$ is independent of the choice of F' and g.

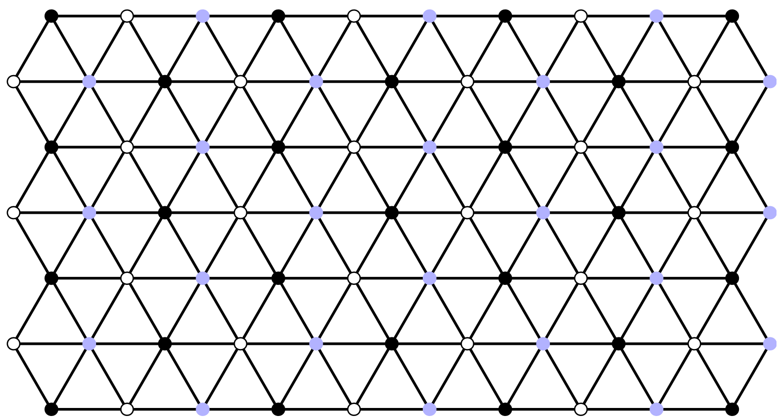
It can be shown (but this is not part of the exercise) that r is distance decreasing (hence, continuous) [6, p. 152].

Exercise ([6, (∗∗) on p. 153])**.** Show X_p is a CAT(0) space. (In particular, X_p is contractible.)
Hint: You need to show that if $x_1, x_2, x_3 \in X_p$ and $p_1, p_2, p_3 \in \mathbb{R}^2$, such that $d(x_i, x_j) = d(p_i, p_j)$ for all i, j, then $d(x_1, x) \le d(p_1, p)$, where x and p are the midpoints of the geodesic segments $\overline{x_2\,x_3}$ and $\overline{p_2\,p_3}$, respectively. To prove this, use a (distance-decreasing) retraction to move the triangle x_1, x_2, x_3 into a flat.

Corollary ([6, p. 161])**.** *Every compact subgroup K of G_p has a fixed point in X_p.*

Proof. In a CAT(0) space, every bounded set has a well-defined circumcenter. Since any K-orbit is a K-invariant, bounded set, the circumcenter of this orbit is a K-invariant point. □

Example (Bruhat-Tits building of $\mathrm{SL}_3(\mathbb{Q}_p)$)**.** The maximal flats in the Bruhat-Tits building of $\mathrm{SL}_3(\mathbb{Q}_p)$ are Euclidean planes, tiled by equilateral triangles.



More generally, the maximal flats in the Bruhat-Tits building of $\mathrm{SL}_n(\mathbb{Q}_p)$ are $(n-1)$-dimensional Euclidean spaces, tiled by regular simplexes. (The n vertices of each simplex are representatives of the orbits of G_p on the 0-skeleton of X_p.) As always, every boundary hyperplane is a branch locus.

It is well known that $\mathrm{SO}(n)$ is the only maximal compact subgroup of $\mathrm{SL}_n(\mathbb{R})$. More generally, the maximal compact subgroups of any connected Lie group are all conjugate to each other. This is not true for p-adic groups:

Proposition. $\mathrm{SL}_n(\mathbb{Q}_p)$ *has exactly n conjugacy classes of maximal compact subgroups.*

Proof. More precisely, every maximal compact subgroup is conjugate to a subgroup of the form

$$\begin{bmatrix} \boxed{\mathbb{Z}_p}_{k\times k} & \boxed{p^{-1}\mathbb{Z}_p}_{k\times(n-k)} \\ \boxed{p\mathbb{Z}_p}_{(n-k)\times k} & \boxed{\mathbb{Z}_p}_{(n-k)\times(n-k)} \end{bmatrix},$$

for some $k \in \{1, 2, \ldots, n\}$. (When $k = n$, this group is $\mathrm{SL}_n(\mathbb{Z}_p)$.)

Let K be a maximal compact subgroup of G_p. Since K is compact, it must fix some point in X_p. Since G_p is type-preserving, this implies that K fixes a vertex v of the simplicial complex. Therefore, K is contained in the stabilizer of v. The stabilizer is compact (since G_p acts properly on X_p), so maximality implies that K equals the stabilizer of v. However, there are only n types of vertices, all vertices of the same type are in the same G_p-orbit, and stabilizers of points in the same orbit are conjugate. Therefore, there are at most n conjugacy classes of maximal compact subgroups. □

Remark. The n subgroups listed at the start of the above proof are the stabilizers of the vertices of a particular simplex in X_p. They (and their conjugates) are the “maximal parahoric subgroups” of $\mathrm{SL}_n(\mathbb{Q}_p)$. (It is interesting to note that they are all conjugate under $\mathrm{GL}_n(\mathbb{Q}_p)$, even though they are not conjugate by any matrix of determinant 1.)

Remark. In general, the same proof shows that the number of conjugacy classes of maximal compact subgroups of G_p is $\operatorname{rank}_{\mathbb{Q}_p} G_p$.

Lecture 3. More applications of Bruhat-Tits buildings

3.1. **Arithmeticity.** A special case of an amazing theorem of G. A. Margulis [12] (or see [13, Thm. 9.1.11, p. 298] or [19, Thm. 6.1.2, p. 114]) tells us that if $G = \mathrm{SL}_n(\mathbb{R})$, with $n \geq 3$, then every cocompact, discrete subgroup Γ of G is “arithmetic.” This makes it possible to provide a quite explicit list of all the possibilities for Γ. More generally, Margulis’ theorem applies whenever G is a connected, simple real Lie group that contains a 2-dimensional group of diagonal matrices. By using Bruhat-Tits buildings, Gromov and Schoen [9] were able to prove this arithmeticity theorem for $G = \mathrm{Sp}(1, n)$, which has only a 1-dimensional group of diagonal matrices.

Here is a highly simplified (and inaccurate) outline of the proof. Embed G in some $\mathrm{SL}_N(\mathbb{R})$. To show that Γ is arithmetic, we need to show that $\Gamma \subset G \cap \mathrm{SL}_N(\mathbb{Z})$ (after a change of basis). The vanishing of a certain cohomology group $H^1(G; \mathfrak{g})$ implies that $\Gamma \subset G \cap \mathrm{SL}_N(\mathbb{Q})$. Thus, for every prime p, we have an embedding of Γ in $\mathrm{SL}_N(\mathbb{Q}_p)$. So Γ acts on the Bruhat-Tits building X_p. It also acts on the symmetric space $X_\infty = G/K$ associated to $\mathrm{Sp}(1, n)$.

By choosing a Γ-equivariant map from X_∞ to X_p that minimizes a certain energy functional, Gromov and Schoen obtain a Γ-equivariant map from X_∞ to X_p that is harmonic. (To achieve this, they developed the necessary theory of harmonic maps into (non-manifold) simplicial complexes.) It can be shown that any such harmonic map must be constant. Since this constant map is Γ-equivariant, the group Γ must have a fixed point in X_p. So Γ is contained in the stabilizer of a point, which is a compact subgroup of $\mathrm{SL}_N(\mathbb{Q}_p)$. This means that the prime p occurs to only a bounded power in the denominator of any element of Γ. Then a finite-index subgroup Γ' has the property that no power of p occurs in any denominator. Repeating the argument to eliminate all primes from the denominators yields a finite-index subgroup Γ'' that is contained in $\mathrm{SL}_N(\mathbb{Z})$, as desired.

3.2. **Decompositions of G_p.** The Cartan decomposition $G = KAK$ and Iwasawa decomposition $G = KAN$ are well known for $\mathrm{SL}_n(\mathbb{R})$ and other simple Lie groups. There are analogous results for $\mathrm{SL}_n(\mathbb{Q}_p)$ (and other p-adic groups).

Theorem (Bruhat-Tits [8, (4.4.3)]). *Let* $G_p = \mathrm{SL}_n(\mathbb{Q}_p)$, $K = \mathrm{SL}_n(\mathbb{Z}_p)$, $A = \{$ *diagonal matrices* $\}$, $N = \{$*upper triangular with* 1*'s on diagonal*$\}$. *Then:*

(1) (*Cartan decomposition*): $G_p = KAK$.
(2) (*Iwasawa decomposition*): $G_p = KAN$.

Proof of the Cartan decomposition. Let $h \in G_p$, let F be the maximal flat in X_p that is fixed by A, and let $x \in F$ be a basepoint.

Since X_p is a building, there is a maximal flat F' that contains both x and hx. Furthermore, there exists $g \in G_p$, such that $gF' = F$ (and g fixes $F' \cap F$ pointwise, so $gx = x$). Then $ghx \in F$, and is the same type as x, so there exists $a \in A$, such that $a(ghx) = x$. Hence, g and agh both belong to $\mathrm{Stab}_{G_p}(x) = K$, so $h = g^{-1}a^{-1}(agh) \in KAK$. □

3.3. **Recent results.** Fundamental properties of S-arithmetic groups are proved by using the Tits building. Here are a couple of important-sounding examples from the internet:

L. Ji: Large scale geometry, compactifications and the integral Novikov conjectures for arithmetic groups, in *Third International Congress of Chinese Mathematicians*. Amer. Math. Soc., Providence, 2008, pp. 317–344. MR2409642, `http://www.cms.zju.edu.cn/UploadFiles/AttachFiles/2006925225424691.pdf`

H. Rüping: The Farrell-Jones conjecture for S-arithmetic groups (preprint). `http://arxiv.org/abs/1309.7236`

3.4. **Explicit construction of the Bruhat-Tits building of $\mathrm{SL}_n(\mathbb{Q}_p)$.**

Definition (cf. `https://en.wikipedia.org/wiki/Building_(mathematics)`).

- If $\{v_1, \ldots, v_n\}$ is any basis of $(\mathbb{Q}_p)^n$, then $\mathbb{Z}_p v_1 + \cdots + \mathbb{Z}_p v_n$ is called a *lattice* in $(\mathbb{Q}_p)^n$.
- Equivalence relation: $\Lambda_1 \sim \Lambda_2$ if there exists $k \in \mathbb{Z}$, such that $p^k \Lambda_1 = \Lambda_2$.
- Each equivalence class is a vertex of X_p.
- If $p\Lambda_k \subsetneq \Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_k$, then $[\Lambda_0], \ldots, [\Lambda_k]$ are the vertices of a k-simplex in X_p.

The group A of diagonal matrices fixes the maximal flat whose vertices are represented by lattices of the form $p^{k_1}\mathbb{Z}_p \oplus p^{k_2}\mathbb{Z}_p \oplus \cdots \oplus p^{k_n}\mathbb{Z}_p$.

Exercise.

(1) Show (directly) that A has n orbits on the vertices of this maximal flat.
(2) Calculate the stabilizer of a vertex in each orbit.

References

[1] P. Abramenko and K. S. Brown: *Buildings: Theory and Applications*. Springer, New York, 2008. MR2439729, ISBN 978-0-387-78834-0

[2] R. C. Alperin and P. B. Shalen: Linear groups of finite cohomological dimension, *Invent. Math.* 66 (1982), no. 1, 89–98. MR0652648

[3] A. Borel and J.-P. Serre: Corners and arithmetic groups, *Comment. Math. Helv.* 48 (1973), 436–491. MR0387495

[4] A. Borel and J.-P. Serre: Cohomologie d'immeubles et de groupes S-arithmétiques, *Topology* 15 (1976), no. 3, 211–232. MR0447474

[5] K. Brown: *Cohomology of Groups*. Springer, New York, 1982. MR0672956, ISBN 0-387-90688-6

[6] K. Brown: *Buildings*. Springer, New York, 1989. MR0969123, ISBN 0-387-96876-8. http://www.math.cornell.edu/~kbrown/buildings/brown1988.pdf

[7] F. Bruhat and J. Tits: Groupes algébriques simples sur un corps local, in T. A. Springer, ed., *Proceedings of a Conference on Local Fields (Driebergen, 1966)* Springer, Berlin, 1967, pp. 23–36. MR0230838

[8] F. Bruhat and J. Tits: Groupes réductifs sur un corps local, *Inst. Hautes Études Sci. Publ. Math.* 41 (1972) 5–251. MR0327923

[9] M. Gromov and R. Schoen: Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one, *Inst. Hautes Études Sci. Publ. Math.* 76 (1992) 165–246. MR1215595, http://www.numdam.org/item?id=PMIHES_1992__76__165_0

[10] Y. Ihara: On discrete subgroups of the two by two projective linear group over $\mathfrak{p}$-adic fields, *J. Math. Soc. Japan* 18 (1966) 219–235. MR0223463

[11] L. Ji: Buildings and their applications in geometry and topology, *Asian J. Math.* 10 (2006), no. 1, 11–80. MR2213684, http://www.math.lsa.umich.edu/~lji/survey.pdf

[12] G. A. Margulis: Arithmeticity of the irreducible lattices in the semi-simple groups of rank greater than 1, *Invent. Math.* 76 (1984) 93–120. MR0739627

[13] G. A. Margulis: *Discrete Subgroups of Semisimple Lie Groups.* Springer, Berlin Heidelberg New York, 1991. MR1090825

[14] M. S. Raghunathan: *Discrete Subgroups of Lie groups.* Springer, New York, 1972. MR507234

[15] M. Ronan: *Lectures on Buildings*. University of Chicago Press, Chicago, 2009. MR2560094, ISBN 0-226-72499-9

[16] J.–P. Serre: Cohomologie des groupes discrets. Springer Lecture Notes in Math. #244, 1971, pp. 337–350. MR0422504

[17] C. L. Siegel: *Zur Reduktionstheorie Quadratischer Formen.* Mathematical Society of Japan, Tokyo, 1959. MR0140486

[18] J. Tits: Reductive groups over local fields, in *Automorphic Forms, Representations and L-functions (Corvallis, 1977)*, Amer. Math. Soc., Providence, R.I., 1979, Part 1, pp. 29–69. MR0546588

[19] R. J. Zimmer: *Ergodic Theory and Semisimple Groups*. Birkhäuser, Boston, 1984. MR0776417