LATTICES IN SL(3, \mathbb{R})
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Abstract. SL(3, \mathbb{Z}) is the easiest example of an arithmetic lattice in SL(3, \mathbb{R}), because SL(3, \mathbb{Q}) is the obvious \mathbb{Q}-form of SL(3, \mathbb{R}). We will see how to construct all of the other lattices, by finding all of the possible \mathbb{Q}-forms. They come from either division algebras or unitary groups, or a combination of the two.

1. Statement of a main result

Every point in \mathbb{R}^n is a bounded distance from some point with integer coordinates, so modding out the integer points of \mathbb{R}^n yields a compact manifold; namely, \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n is the n-torus, which is compact. On the other hand, modding out the integer points of the Lie group SL(3, \mathbb{R}) does not yield a compact space. However, the quotient is “almost” compact: for \( G = \text{SL}(3, \mathbb{R}) \) and \( \Gamma = \text{SL}(3, \mathbb{Z}) \), it is well known that \( G/\Gamma \) has finite volume. Because of this (and since \( \Gamma \) is discrete), we say that \( \Gamma \) is a lattice in \( G \). Many other Lie groups also have the property that their integer points form a lattice:

Theorem (Borel & Harish-Chandra, 1962). Suppose \( G \) is a connected subgroup of SL(n, \mathbb{R}), for some n. If \( G \) is perfect (i.e., \([G, G] = G\)), then the following are equivalent:

(1) \( G_{\mathbb{Z}} \) is a lattice in \( G \), where \( G_{\mathbb{Z}} = G \cap \text{SL}(n, \mathbb{Z}) \).

(2) \( G_{\mathbb{Q}} \) is dense in \( G \), where \( G_{\mathbb{Q}} = G \cap \text{SL}(n, \mathbb{Q}) \).

(3) \( G \) is defined over \( \mathbb{Q} \). This means that \( G \) is defined by polynomial equations with rational coefficients. More precisely, there are polynomials

\[
\begin{align*}
  f_1(x_{1,1}, \ldots, x_{n,n}), \ldots, f_k(x_{1,1}, \ldots, x_{n,n})
\end{align*}
\]

with coefficients in \( \mathbb{Q} \), such that

\[
G \doteq \{ (g_{i,j}) \in \text{SL}(n, \mathbb{R}) \mid f_k(g_{1,1}, \ldots, g_{n,n}) = 0, \text{ for all } k \}.
\]

Remark. We ignore finite groups throughout this lecture: writing \( G \doteq H \) means that some finite-index subgroup of \( G \) is equal to some finite-index subgroup of \( H \). In other words, \( G \) and \( H \) are commensurable.

Corollary. \( \text{SO}(1, n)_{\mathbb{Z}} \) is a lattice in \( \text{SO}(1, n) \).
Proof. By definition, \( \text{SO}(1,n) = \{ g \in \text{SL}(n+1, \mathbb{R}) \mid gI_{1,n}g^T = I_{1,n} \} \), where

\[
I_{1,n} = \begin{bmatrix}
1 & & \\
-1 & 1 & \\
& -1 & \ddots & \\
& & & -1
\end{bmatrix}.
\]

Write out the equation \( gI_{1,n}g^T = I_{1,n} \) in terms of the matrix entries \((g_{i,j})\). This yields \((n+1)^2\) polynomial equations, with coefficients in \(\mathbb{Q}\). Therefore \(\text{SO}(1,n)\) is defined over \(\mathbb{Q}\), so the theorem tells us that \(\text{SO}(1,n)_\mathbb{Z}\) is a lattice.

We have a special name for lattices that are constructed by taking the integral points of \(G\):

**Definition.** If \(\Gamma \cong G_\mathbb{Z}\), then \(\Gamma\) is an **arithmetic** lattice in \(G\).

**Remark.** Actually, for cocompact lattices, the definition should be generalized somewhat, by allowing compact factors to be added to \(G\).

A typical Lie group \(G\) has *many* arithmetic lattices, because there are lots of different ways to embed \(G\) in \(\text{SL}(n, \mathbb{R})\) (if \(n\) is large), and different embeddings can yield very different lattices. The subgroup \(G_\mathbb{Q}\) is called a “\(\mathbb{Q}\)-form” of \(G\). Finding all the arithmetic lattices in \(G\) (up to commensurability) is the same as finding all the \(\mathbb{Q}\)-forms of \(G\) (up to isomorphism).

Arithmetic lattices are the “obvious” lattices, and are fairly well understood. In particular, these lectures will present a list of all the arithmetic lattices in \(\text{SL}(3, \mathbb{R})\) (and a similar list has been made for all other simple Lie groups.) (In contrast, the non-arithmetic lattices are not all known in the cases where they exist, except in \(\text{SL}(2, \mathbb{R})\).) In many cases, these obvious lattices are the only ones:

**Theorem** (Margulis Arithmeticity Theorem, 1975). Every lattice in \(G = \text{SL}(n, \mathbb{R})\) is arithmetic if \(n \geq 3\).

More generally, if \(G\) is simple and \(\text{rank}_\mathbb{R}G \geq 2\), then every lattice in \(G\) is arithmetic.

Thus, to find all of the lattices in \(\text{SL}(3, \mathbb{R})\), we just need to find all of its \(\mathbb{Q}\)-forms. One of the \(\mathbb{Q}\)-forms is \(\text{SL}(3, \mathbb{Q})\), which corresponds to the lattice \(\text{SL}(3, \mathbb{Z})\). Before describing the others, let us briefly recall the special unitary groups.

**Example.** We have

\[
\text{SU}(m,n) = \{ g \in \text{SL}(m+n, \mathbb{C}) \mid Q(g\overline{z}) = Q(\overline{z}) \}
\]

where \(Q(\overline{z}) = \left| z_1 \right|^2 + \cdots + \left| z_m \right|^2 - \left| z_{m+1} \right|^2 - \cdots - \left| z_{m+n} \right|^2\)

\[
= \{ g \in \text{SL}(m+n, \mathbb{C}) \mid gI_{m,n}g^T = I_{m,n} \}.
\]
This construction relies on complex conjugation, which is the nontrivial element of the Galois group \( \text{Gal}(\mathbb{C}/\mathbb{R}) \).

Unitary groups can be constructed in an analogous way for any quadratic field extension \( F/K \), replacing complex conjugation with the Galois automorphism \( \sigma \in \text{Gal}(F/K) \). To construct arithmetic lattices, we take \( K = \mathbb{Q} \):

**Proposition.** Let

- \( F \) be a real quadratic extension of the field \( \mathbb{Q} \) (this means \( F = \mathbb{Q}(\sqrt{r}) \), where \( r \) is a square-free integer that is greater than 1),
- \( \sigma \) be the Galois automorphism of \( F \), which means \( \sigma(a + b\sqrt{r}) = a - b\sqrt{r} \),
- \( J = \begin{bmatrix} 1 & 1 \\
1 & 1 \end{bmatrix} \) or you could use \( J = \begin{bmatrix} 1 & -1 \\
-1 & 1 \end{bmatrix} \), and
- \( \Gamma_r = \text{SU}(J, \sigma; \mathbb{Z}[\sqrt{r}]) = \left\{ g \in \text{SL}(3, \mathbb{Z}[\sqrt{r}]) \mid gJ(g^\sigma)^T = J \right\} \).

Then \( \Gamma_r \) is an arithmetic lattice in \( \text{SL}(3, \mathbb{R}) \).

**Theorem** (Weil 1960 (or Siegel earlier?)). Suppose \( \Gamma \) is an (arithmetic) lattice in \( G = \text{SL}(3, \mathbb{R}) \). If \( G/\Gamma \) is not compact, then \( \Gamma \) is commensurable either to \( \text{SL}(3, \mathbb{Z}) \) or to \( \Gamma_r \), for some \( r \).

**Remark.** The value of \( r \) is unique: if \( r_1 \neq r_2 \), then \( \Gamma_{r_1} \) is not abstractly commensurable to \( \Gamma_{r_2} \). This means that no finite-index subgroup of \( \Gamma_{r_1} \) is isomorphic to a finite-index subgroup of \( \Gamma_{r_2} \).

2. WHY IS \( \Gamma_r \) AN ARITHMETIC LATTICE?

Let \( G = \text{SL}(3, \mathbb{R}) \). We show there is an embedding \( \rho : G \to \text{SL}(6, \mathbb{R}) \), such that

\[
\rho(G) \cap \text{SL}(6, \mathbb{Z}) = \rho(\Gamma_r).
\]

Begin by observing that there is an obvious copy of \( G \times G \) in \( \text{SL}(6, \mathbb{R}) \):

\[
G \times G = \begin{bmatrix}
* & * & * \\
* & * & * \\
* & * & * \\
* & * & * \\
* & * & * \\
* & * & *
\end{bmatrix}.
\]

Let

\[
\hat{G} = \{ (g, h) \in G \times G \mid gh^T = J \}.
\]

Since any choice of \( g \) determines a unique \( h \), we see that \( \hat{G} \cong G \). So we may let \( \rho(G) = \hat{G} \). Then \( \rho(\Gamma_r) = \{ (g, g^\sigma) \mid g \in \Gamma_r \} \).
Note that the vectors $(1, 1)$ and $(\sqrt{r}, -\sqrt{r})$ are linearly independent, so they form a basis of $\mathbb{R}^2$. Thus, there is an invertible linear transformation of $\mathbb{R}^6$ that maps $\mathbb{Z}^6$ to

$$\Lambda = \{(x, y, z, x^\sigma, y^\sigma, z^\sigma) \mid x, y, z \in \mathbb{Z}[^\sqrt{r}]\}.$$ 

Since $\text{SL}(6, \mathbb{Z}) = \{a \in \text{SL}(6, \mathbb{R}) \mid a\mathbb{Z}^6 = \mathbb{Z}^6\}$, this means that, after a change of basis, we may pretend that

$$(G \times G) \cap \text{SL}(6, \mathbb{Z}) = \{(g, h) \in G \times G \mid (g, h)\Lambda = \Lambda\} = \{(g, g^\sigma) \mid g \in \text{SL}(3, \mathbb{Z}[\sqrt{r}])\}.$$

Then

$$\rho(G) \cap \text{SL}(6, \mathbb{Z}) = \hat{G} \cap ((G \times G) \cap \text{SL}(6, \mathbb{Z}))$$

$$= \left\{(g, g^\sigma) \mid g \in \text{SL}(3, \mathbb{Z}[\sqrt{r}]), \begin{array}{l} gJ(g^\sigma)^T = J \end{array} \right\}$$

$$= \rho(\Gamma_r),$$

as desired.

Now, in order to conclude that $\Gamma_r$ is an arithmetic lattice, all that remains is to show that $\rho(G) = \hat{G}$ is defined over $\mathbb{Q}$ (with respect to this basis). This can be done directly, by finding appropriate polynomials with rational coefficients, but this may be confusing, since we are working in a strange basis.

Instead, we will verify the equivalent condition that $\hat{G}_Q$ is dense in $\hat{G}$. Note that

$$G_Q := \rho^{-1}(\hat{G}_Q) = \left\{ g \in \text{SL}(3, \mathbb{Q}[\sqrt{r}]) \mid gJ(g^\sigma)^T = J \right\}.$$

Then a simple calculation shows that

$$\left\{ \begin{bmatrix} 1 & x & t\sqrt{r} - \frac{x x^\sigma}{2} \\ \sqrt{r} & 1 & -x^\sigma \\ 1 & \end{bmatrix} \begin{array}{l} x \in \mathbb{Q}[\sqrt{r}], \\ t \in \mathbb{Q} \end{array} \right\} \subseteq G_Q.$$

This subgroup is dense in

$$U = \begin{bmatrix} 1 & * & * \\ * & 1 & * \\ * & & 1 \end{bmatrix}.$$

Similarly, $G_Q$ also contains a dense subgroup of $U^T$. (In fact, it is easy to verify that $(G_Q)^T = G_Q$.) Since $\langle U, U^T \rangle = \text{SL}(3, \mathbb{R}) = G$, we conclude that $G_Q$ is dense in $G$, as desired.


To be continued...

(Calculating a cohomology group shows these are the only noncocompact lattices.)
3. Why are these the only (noncocompact) lattices in $\text{SL}(3, \mathbb{R})$?

Recall.
- Every lattice $\Gamma$ in $G = \text{SL}(3, \mathbb{R})$ is arithmetic. [Margulis Arithmeticity Theorem]
- If $G/\Gamma$ is not compact, this means there is an embedding $G \hookrightarrow \text{SL}(N, \mathbb{R})$, such that
  - $\Gamma \trianglelefteq G_\mathbb{Z} = G \cap \text{SL}(N, \mathbb{Z})$, and
  - $G$ is defined over $\mathbb{Q}$ (defined by polynomial equations with coefficients in $\mathbb{Q}$).
- We claim this implies $\Gamma$ is either $\text{SL}(3, \mathbb{Z})$ or $\Gamma_r$ (modulo finite groups), where $\Gamma_r = \text{SU}(J, \sigma; \mathbb{Z}[\sqrt{r}]) = \{ g \in \text{SL}(3, \mathbb{Z}[\sqrt{r}]) \mid gJ(g^\sigma)^T = J \}$.
- The arithmetic group $G_\mathbb{Z}$ is determined (modulo finite groups), by the "$\mathbb{Q}$-form" $G_\mathbb{Q}$. Thus, we wish to show that $G_\mathbb{Q}$ must be either $\text{SL}(3, \mathbb{Q})$ or $\text{SU}(J, \sigma; \mathbb{Q}[\sqrt{r}])$.

We will explain how to determine all of the possible $\mathbb{Q}$-forms of $\text{SL}(n, \mathbb{R})$. Some of them do not arise when $n = 3$ and $G/G_\mathbb{Z}$ is not compact, leaving only the above two possibilities in this case.

**How to find the $\mathbb{Q}$-forms of $\text{SL}(n, \mathbb{R})$ by using group cohomology**

Let $G = \text{SL}(n, \mathbb{R})$, and suppose we have an embedding $\rho: G \rightarrow \text{SL}(N, \mathbb{R})$, such that $\rho(G)$ is defined over $\mathbb{Q}$. To figure out what $\rho(G)_\mathbb{Q}$ could be, we take an algebraic approach.

Galois theory tells us that
$$\mathbb{Q} = \{ z \in \mathbb{C} \mid \sigma(z) = z \text{ for all } \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \}.$$ 

Therefore
$$\text{SL}(N, \mathbb{Q}) = \{ g \in \text{SL}(N, \mathbb{C}) \mid \sigma(g) = g \text{ for all } \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \},$$

so
$$\rho(G)_\mathbb{Q} = \rho(G) \cap \text{SL}(N, \mathbb{Q}) = \{ g \in \rho(G) \mid \sigma(g) = g \text{ for all } \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \}.$$ 

**Important fact:** The matrix entries of $\rho$ are polynomial functions (with coefficients in $\mathbb{R}$), and essentially the same is true for $\rho^{-1}$.

Therefore $\rho$ extends to $\rho: \text{SL}(n, \mathbb{C}) \xrightarrow{\cong} \rho(G)_\mathbb{C}$.

Since $\rho(G)$ is defined over $\mathbb{Q}$, we know that $\rho(G)_\mathbb{C}$ is invariant under $\text{Gal}(\mathbb{C}/\mathbb{Q})$, so, for $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$ we have
$$G_\mathbb{C} \xrightarrow{\rho} \rho(G)_\mathbb{C} \xrightarrow{\sigma} \rho(G)_\mathbb{C} \xrightarrow{\rho^{-1}} G_\mathbb{C}.$$ 

Let $\tilde{\sigma} = \rho^{-1} \sigma \rho: G_\mathbb{C} \rightarrow G_\mathbb{C}$ be the composition, so
$$G_\mathbb{Q} = \{ g \in G_\mathbb{C} \mid \tilde{\sigma}(g) = g \text{ for all } \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \}.$$
Now let 

$$\alpha_\sigma = \tilde{\sigma} \sigma^{-1} : G_\mathbb{C} \to G_\mathbb{C}.$$ 

It is easy to see that

- $\alpha_\sigma$ is an automorphism of $G_\mathbb{C}$ (as an abstract group), and
- $\alpha_\sigma$ is continuous (in fact, it’s a polynomial, since $\alpha_\sigma = \rho^{-1} \sigma \rho \sigma^{-1} = \rho^{-1} \sigma \rho$).

So $\alpha_\sigma \in \text{Aut}(G_\mathbb{C})$.

Furthermore, since $\alpha_\sigma = \rho^{-1} \sigma \rho$ is formally a 1-coboundary (as a function of $\sigma$), it is easily seen to be a 1-cocycle of group cohomology, and therefore defines an element of $H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{Aut}(G_\mathbb{C}))$. In fact, this construction provides a one-to-one correspondence between $H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{Aut}(G_\mathbb{C}))$ and the set of $\mathbb{Q}$-forms. Thus:

**Finding all of the arithmetic lattices in $G$ amounts to calculating the “Galois cohomology group” $H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{Aut}(G_\mathbb{C}))$.**

The above discussion is an example of a fairly general principle that if $X$ is an algebraic object that is defined over $\mathbb{Q}$, then $H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{Aut}(X_\mathbb{C}))$ is in one-to-one correspondence with the set of $\mathbb{Q}$-isomorphism classes of $\mathbb{Q}$-defined objects whose $\mathbb{C}$-points are isomorphic to $X_\mathbb{C}$.

**Example.** Suppose $V_1$ and $V_2$ are two vector spaces over $\mathbb{Q}$, and they are isomorphic over $\mathbb{C}$. (I.e., $V_1 \otimes \mathbb{C} \cong V_2 \otimes \mathbb{C}$.) Then the two vector spaces have the same dimension, so they are isomorphic over $\mathbb{Q}$. Thus, the $\mathbb{Q}$-form of any vector space is unique, so $H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{Aut}(V_\mathbb{C})) = 0$, for any complex vector space $V_\mathbb{C}$. In other words, we have $H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{GL}(n, \mathbb{C})) = 0$.

A modification of this argument shows $H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{SL}(n, \mathbb{C})) = 0$.

**Constructing explicit $\mathbb{Q}$-forms from cohomology classes**

It is known that the outer automorphism group of $G_\mathbb{C} = \text{SL}(n, \mathbb{C})$ has only one nontrivial element, namely, the “transpose-inverse” automorphism, defined by $\omega(g) = (g^T)^{-1}$. So

$$\text{Aut}(G_\mathbb{C}) = \text{PSL}(n, \mathbb{C}) \rtimes \langle \omega \rangle.$$ 

Given $\alpha \in H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{Aut}(G_\mathbb{C}))$, corresponding to a $\mathbb{Q}$-form $G_\mathbb{Q}$, we consider two cases.

**Case 1. Assume the image of $\alpha$ is not contained in $\text{PSL}(n, \mathbb{C})$.** Since the action of $\text{Gal}(\mathbb{C}/\mathbb{Q})$ on $\text{Out}(G_\mathbb{C}) \cong \mathbb{Z}_2$ is trivial, the 1-cocycle $\alpha$ induces a nontrivial homomorphism $\overline{\alpha}: \text{Gal}(\mathbb{C}/\mathbb{Q}) \to \text{Out}(G_\mathbb{C})$. Therefore, the kernel of $\overline{\alpha}$ is a subgroup of index 2 in the Galois group, so its fixed field is a quadratic extension $\mathbb{Q}[\sqrt{r}]$ of $\mathbb{Q}$.

Consider any $g \in G_\mathbb{Q}$. For simplicity, let us assume that $\alpha$ is trivial on the kernel of $\overline{\alpha}$. (In Case 2 of the proof, we will see the effect of a nontrivial cocycle
into \( \text{PSL}(n, \mathbb{C}) \).) This means that, for all \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}[\sqrt{\tau}]) \), we have \( \sigma \in \ker \alpha \), so \( \sigma = \tilde{\sigma} \). Therefore

\[
g^\sigma = g^{\tilde{\sigma}} = g.
\]

Since this holds for all \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}[\sqrt{\tau}]) \), we conclude that \( g \in \text{SL}(n, \mathbb{Q}[\sqrt{\tau}]) \).

Now, for \( \sigma \in \text{Gal}(\mathbb{Q}[\sqrt{\tau}]/\mathbb{Q}) \), we have \( \alpha_\sigma = (\text{conj by } A) \omega \) for some \( A \in \text{GL}(n, \mathbb{R}) \), so

\[
g = \tilde{\sigma}(g) = \alpha_\sigma(g) = A \omega \sigma(g) A^{-1} = A ((\sigma g)^T)^{-1} A^{-1},
\]

which means \( g A (\sigma g)^T = A \). I.e., \( g \in \text{SU}(A, \sigma; \mathbb{Q}[\sqrt{\tau}]) \).

One can show that if \( n = 3 \) and \( G/\Gamma \) is not compact, then, after a change of variables, we have \( A = J \), so \( G_{\mathbb{Z}} = \Gamma_{\mathbb{R}} \).

**Case 2.** Assume \( \alpha \in H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{PSL}(n, \mathbb{C})) \). We have a short exact sequence

\[
0 \to \mu_n \to \text{SL}(n, \mathbb{C}) \to \text{PSL}(n, \mathbb{C}) \to 0
\]

where \( \mu_n = Z(\text{SL}(n, \mathbb{C})) \) is the group of \( n \)-th roots of unity in \( \mathbb{C} \). This yields a long exact sequence of cohomology:

\[
H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{SL}(n, \mathbb{C})) \to H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{PSL}(n, \mathbb{C})) \xrightarrow{\delta} H^2(\text{Gal}(\mathbb{C}/\mathbb{Q}), \mu_n).
\]

We have seen that the first term is 0, and Algebraic Number Theorists will recognize the last term as part of the Brauer group, namely, it is the set of central simple algebras of degree \( n \) over \( \mathbb{Q} \).

However, let us approach this calculation in a different way. Every \( \mathbb{C} \)-linear automorphism of the matrix algebra \( \text{Mat}_{n \times n}(\mathbb{C}) \) is inner — conjugation by a matrix in \( \text{GL}(n, \mathbb{C}) \). Since the center acts trivially, this means \( \text{Aut}(\text{Mat}_{n \times n}(\mathbb{C})) = \text{PSL}(n, \mathbb{C}) \). Therefore,

\[
H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{PSL}(n, \mathbb{C})) = H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{Aut}(\text{Mat}_{n \times n}(\mathbb{C}))),
\]

so, by the general principle, it is the set of \( \mathbb{Q} \)-forms of \( \text{Mat}_{n \times n}(\mathbb{C}) \). More precisely, it is the set of algebras \( A \) over \( \mathbb{Q} \), such that \( A \otimes \mathbb{C} \cong \text{Mat}_{n \times n}(\mathbb{C}) \). Such an algebra must be simple, so, by Wedderburn’s Theorem, it is a matrix algebra over a division algebra: \( A \cong \text{Mat}_k(D) \), where \( D \) is a division algebra over \( \mathbb{Q} \) (and the center of \( D \) must be \( \mathbb{Q} \)). The corresponding \( \mathbb{Q} \)-form \( G_\mathbb{Q} \) is \( \text{SL}(k, D) \).

Since \( \dim_\mathbb{Q} A = \dim_\mathbb{C} \text{Mat}_{n \times n}(\mathbb{C}) \), we know that \( k^2d^2 = n^2 \), where \( d^2 \) is the dimension of \( D \) as a vector space over \( \mathbb{Q} \). When \( n = 3 \), this implies \( d \) must be either 1 or 3.

- For \( d = 1 \), we have \( D = \mathbb{Q} \) and \( k = 3 \), so \( G_\mathbb{Q} = \text{SL}(3, \mathbb{Q}) \).
- For \( d = 3 \), we have \( n = 1 \), so \( G_\mathbb{Q} = \text{SL}(1, D) \). This yields a cocompact lattice in \( \text{SL}(3, \mathbb{R}) \), so it does not appear on the list of non-cocompact lattices.
Intermediate case. We seem to have shown that all (arithmetic) lattices in SL(n, R) can be constructed from either unitary groups (Case 1) or division algebras (Case 2). However, the discussion in Case 1 assumes that the restriction of \( \alpha \) to the kernel of \( \overline{\alpha} \) is trivial. If we remove this restriction, then, by the argument of Case 2, the cocycle from \( \text{Gal}(\mathbb{C}/\mathbb{Q}[\sqrt{r}]) \) into \( \text{PSL}(n, \mathbb{C}) \) yields a simple algebra \( \text{Mat}_k(D) \) whose center is \( \mathbb{Q}[\sqrt{r}] \). The Galois automorphism \( \sigma \) of \( \mathbb{Q}[\sqrt{r}] \) can be extended to an anti-automorphism \( \hat{\sigma} \) of \( D \). Then, for some \( A \in \text{Mat}_k(D) \), the corresponding \( \mathbb{Q} \)-form is

\[
G_\mathbb{Q} = \text{SU}(A, \hat{\sigma}; D) = \left\{ g \in \text{SL}(k, D) \mid gA(g^\hat{\sigma})^T = A \right\}.
\]

Note that this \( \mathbb{Q} \)-form is obtained by combining unitary groups with division algebras.

All non-cocompact (arithmetic) lattices in SL(n, R) are one of these, constructed from either unitary groups, or division algebras, or a combination of the two.

Cocompact lattices. All cocompact lattices can also be obtained from the above methods, if we add one more technique that is familiar from the case of SL(2, R). The key point is that we need to slightly extend the definition of an arithmetic lattice. Namely, instead of requiring \( \Gamma \) to be the integer points of \( G \) itself, it may be necessary to choose a compact group \( K \), such that \( G \times K \) is defined over \( \mathbb{Q} \), and allow \( \Gamma \) to be the projection of \( (G \times K)_Z \) to \( G \). Because of this, \( G_\mathbb{Q} \) can be a unitary group over a (totally real) extension of \( \mathbb{Q} \), rather than over \( \mathbb{Q} \) itself. In other words, we need to consider lattices obtained from “Restriction of Scalars”.

\( \mathbb{C} \) vs. \( \overline{\mathbb{Q}} \). For Galois cohomology, we should really be using the algebraic closure \( \overline{\mathbb{Q}} \) of \( \mathbb{Q} \), instead of \( \mathbb{C} \), and \( H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), G_{\overline{\mathbb{Q}}}) \) is defined to be the natural limit of the groups \( H^1(\text{Gal}(F/\mathbb{Q}), G_{\overline{\mathbb{Q}}}) \), where \( F \) ranges over all finite extensions of \( \mathbb{Q} \).

SL(n, R) vs. SL(n, C). To discuss Galois cohomology, we replaced \( \mathbb{R} \) with the algebraically closed field \( \mathbb{C} \). Thus, some of the groups we found might not be \( \mathbb{Q} \)-forms of \( \text{SL}(n, \mathbb{R}) \) (although we know that their complexification is \( \text{SL}(3, \mathbb{C}) \)). For example, if \( G_\mathbb{Q} = \text{SU}(J, \sigma; \mathbb{Q}[\sqrt{r}]) \), and \( r < 0 \), then \( G_\mathbb{R} \) is \( \text{SU}(2, 1) \), not \( \text{SL}(3, \mathbb{R}) \).

- In practice, one can determine which of the groups we constructed are \( \mathbb{Q} \)-forms of \( \text{SL}(3, \mathbb{R}) \).
- Abstractly, \( \text{SL}(3, \mathbb{R}) \) is a \( \mathbb{R} \)-form of \( \text{SL}(3, \mathbb{C}) \), so, by the general principle, it is represented by a cohomology class \( \beta \in H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), \text{Aut}(G_\mathbb{C})) \). There is a natural restriction homomorphism

\[
r: H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{Aut}(G_\mathbb{C})) \to H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), \text{Aut}(G_\mathbb{C})),
\]

and the \( \mathbb{Q} \)-forms of \( \text{SL}(3, \mathbb{R}) \) are represented by the elements of \( r^{-1}(\beta) \).