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THE NON-ORIENTABLE GENUS OF SOME METACYCLIC GROUPS

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We describe non-orientable, octagonal embeddings for certain 4-valent, bipartite Cayley graphs of finite metacyclic groups, and give a class of examples for which this embedding realizes the non-orientable genus of the group. This yields a construction of Cayley graphs for which $2\gamma - \tilde{\gamma}$ is arbitrarily large, where γ and $\tilde{\gamma}$ are the orientable genus and the non-orientable genus of the Cayley graph.

1. Introduction

It is well-known that if a regular d-valent graph on v vertices and of girth g admits an embedding into an orientable surface of genus γ, then

$$\gamma \geq \left\lceil 1 - \frac{v}{2} + \frac{vd}{4}\left(1 - \frac{2}{g}\right)\right\rceil. \tag{1.1}$$

In the case of a 2-cell embedding (i.e., in the case where every region of the embedding is homeomorphic to a disk), the inequality can easily be derived from the Euler Formula, $v - e + f = 2 - 2\gamma$ (where e is the number of edges and f is the number of 2-cells of the embedding): the Handshaking Lemma asserts $2e = dv$, and applying the Handshaking Lemma to the dual graph yields $2e \geq gf$; incorporating these observations into the Euler Formula and using the fact that γ is an integer results in precisely (1.1). For an argument extending this inequality to the case where not every region is homeomorphic to a disk, see [12].

A similar result is true for an embedding into a non-orientable surface of non-orientable genus $\tilde{\gamma}$ [8, Theorem 2b]. In this case, we use the non-orientable version of the Euler Formula, $v - e + f = 2 - \tilde{\gamma}$, and the conclusion is

$$\tilde{\gamma} \geq \left\lceil 2 - v + \frac{vd}{2}\left(1 - \frac{2}{g}\right)\right\rceil. \tag{1.2}$$

The *genus* of a graph is the minimum of the genera of the orientable surfaces on which the graph can be embedded; the *genus* of a finite group [11], [7], [3] is

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the minimum of the genera of the Cayley graphs of the group. (To find the genus of a finite group, it suffices to consider only irredundant Cayley graphs.) The *non-orientable genus* of a graph or of a group is defined similarly. Most groups whose genus (or non-orientable genus) is known have an irredundant Cayley graph that yields equality in (1.1) (or in (1.2), respectively) and, at the same time, achieves the minimum (among all the irredundant Cayley graphs of the group) for the right-hand side of this inequality. Such groups are said to admit *best* (orientable or non-orientable) embeddings. Groups that do not admit best embeddings are much more difficult to deal with. Here we describe non-orientable 2-cell embeddings for certain Cayley graphs of metacyclic groups, and give a class of groups for which this embedding is a best non-orientable embedding.

For a Cayley graph of orientable genus γ and non-orientable genus $\tilde{\gamma}$, the quantity $2\gamma - \tilde{\gamma}$ is a measure of the difference between the minimal orientable embeddings and the minimal non-orientable embeddings of the Cayley graph [8]. (This quantity compares the Euler characteristic $2 - 2\gamma$ of an orientable surface with the Euler characteristic $2 - \tilde{\gamma}$ of a non-orientable surface.) Every graph satisfies $2\gamma - \tilde{\gamma} \geq -1$ [9, Theorem 7]. There are examples of graphs of non-orientable genus one and arbitrarily large orientable genus [1]. For groups, Brin, Rauschenberg, and Squier [2] have shown that $2\gamma - \tilde{\gamma} = 3$ when Γ is the nonabelian, metacyclic group of order 27. In this paper, we construct a family of Cayley graphs for which $2\gamma - \tilde{\gamma}$ is arbitrarily large; these are the first known Cayley graphs for which $2\gamma - \tilde{\gamma} > 3$. It would be interesting to construct a family of groups for which $2\gamma - \tilde{\gamma}$ is arbitrarily large. The groups constructed in this paper are good candidates, but we have an interesting lower bound on the orientable genus only for certain of their Cayley graphs; we do not know how to prove an interesting lower bound that holds for *all* of the Cayley graphs of the groups.

(3.1′) Proposition. *Suppose $\{x, y\}$ is an irredundant, 2-element generating set for a finite group Γ. If the corresponding Cayley graph, $\mathrm{Cay}(\Gamma; x^{\pm 1}, y^{\pm 1})$, is bipartite and 4-valent, and if the subgroup generated by x is a normal subgroup of Γ, then $\mathrm{Cay}(\Gamma; x^{\pm 1}, y^{\pm 1})$ has an octagonal, non-orientable embedding; therefore, $\tilde{\gamma}(\Gamma) \leq 2 + |\Gamma|/2$.*

(4.1′) Theorem. *Let m, n, and k be powers of 2, with $64 \leq 8k \leq m \leq nk$. Then the embedding of Proposition 3.1 is a best non-orientable embedding for the metacyclic group given by the following generators and relations:*

$$\langle x, y \mid x^m = y^n = e,\ y^{-1}xy = x^{k+1} \rangle.$$

In particular, the non-orientable genus of this group is $2 + mn/2$.

Remark. Corollary 4.2 gives some additional examples where the embedding of Proposition 3.1 is a best non-orientable embedding.

(5.2′) Theorem. *Let m, n, and k be powers of 2, with $256 \leq k^2 < m \leq nk$, and let Γ be the metacyclic group given by the following generators and relations:*

$$\langle x, y \mid x^m = y^n = e,\ y^{-1}xy = x^{k+1} \rangle.$$

Then the Cayley graph $G = \mathrm{Cay}(\Gamma; x^{\pm 1}, y^{\pm 1})$ satisfies $2\gamma(G) - \tilde{\gamma}(G) \geq n/10$.

2. Preliminaries: group theory and Cayley graphs

Definition. A group Γ is *metacyclic* if Γ has a normal subgroup N such that the subgroup N is cyclic, and the quotient group Γ/N is also cyclic.

(2.1) Remark. (see [4, §1.8, pp. 9–10]). Any finite, metacyclic group has a presentation of the form

$$\langle x, y \mid x^m = e,\ y^n = x^r,\ y^{-1}xy = x^k \rangle,$$

for some natural numbers m, n, r, k such that $k^n \equiv 1 \pmod{m}$, $kr \equiv r \pmod{m}$, and $\gcd(k, m) = 1$. Conversely, for any such natural numbers, the given presentation defines a metacyclic group of order mn.

Definition. For any subset Δ of a group Γ, we use $\langle \Delta \rangle$ to denote the subgroup of Γ generated by Δ. We say that Δ is a *generating set* for Γ if $\langle \Delta \rangle = \Gamma$, and that Δ is *symmetric* if, for every $x \in \Delta$, we have $x^{-1} \in \Delta$.

Definition. Suppose Δ is a symmetric generating set for a group Γ, and that Δ does not contain the identity element e of Γ. Then the *Cayley graph* $\operatorname{Cay}(\Gamma; \Delta)$ is a graph defined as follows. The vertices of $\operatorname{Cay}(\Gamma; \Delta)$ are the elements of Γ; for each $g \in \Gamma$ and each $x \in \Delta$, there is an edge joining g and gx.

Definition. A symmetric generating set Δ for a finite group Γ is *irredundant* if $\langle \Delta \setminus \{x, x^{-1}\} \rangle$ is a proper subgroup of Γ, for every $x \in \Delta$. We say that the Cayley graph $\operatorname{Cay}(\Gamma; \Delta)$ is *irredundant* if Δ is an irredundant symmetric generating set.

Definition. [5, p. 173]. The *Frattini subgroup* $\Phi(\Gamma)$ of a finite group Γ is the intersection of all the maximal subgroups of Γ.

One can show (see [5, Theorem 5.1.1(i)]) that an element x of Γ belongs to $\Phi(\Gamma)$ if and only if x belongs to no irredundant symmetric generating set for Γ. The following lemma is another way of saying essentially the same thing.

(2.2) Lemma. [5, Theorem 5.1.1(i)]. *Let* $\operatorname{Cay}(\Gamma; \Delta)$ *be an irredundant Cayley graph of a finite group* Γ, *and let* $\overline{\Gamma} = \Gamma/\Phi(\Gamma)$. *Then* $\operatorname{Cay}(\overline{\Gamma}; \overline{\Delta})$ *is an irredundant Cayley graph of* $\overline{\Gamma}$, *where* $\overline{\Delta}$ *is the image of* Δ *under the natural homomorphism* $\Gamma \to \overline{\Gamma}$. ∎

(2.3) Lemma. [5, Theorem 5.1.3]. *If* Γ *is a finite p-group, then* $\Phi(\Gamma) = \langle \Gamma^p, [\Gamma, \Gamma] \rangle$, *where* $\Gamma^p = \langle x^p \mid x \in \Gamma \rangle$. ∎

(2.4) Lemma. *Let* Γ *and* Λ *be finite groups, and assume* $\gcd(|\Gamma|, |\Lambda|) = 1$. *Then* $\Phi(\Gamma \times \Lambda) = \Phi(\Gamma) \times \Phi(\Lambda)$.

Proof. Since $|\Gamma|$ and $|\Lambda|$ are relatively prime, every subgroup of $\Gamma \times \Lambda$ is of the form $A \times B$, where A is a subgroup of Γ and B is a subgroup of Λ. Therefore, maximal subgroups of $\Gamma \times \Lambda$ are those of the form $M \times \Lambda$ or $\Gamma \times N$, where M is a maximal subgroup of Γ and N is a maximal subgroup of Λ. The conclusion follows. ∎

We need only one direction ($\Rightarrow$) of the following proposition, but we prove the converse because it provides an amusing characterization of 2-groups. A similar result appears in [10, Theorem 2.2].

(2.5) Proposition. *A finite group* Γ *is a 2-group if and only if every irredundant Cayley graph on* Γ *is bipartite.*